

Leibniz algebroid associated with a Nambu-Poisson structure

R. IBÁÑEZ ¹, M. de LEON ², J.C. MARRERO ³ and E. PADRON ³

¹*Departamento de Matemáticas, Facultad de Ciencias,
Universidad del País Vasco,
Apartado 644, 48080 Bilbao, Spain,
mtpibtor@lg.ehu.es*

²*Instituto de Matemáticas y Física Fundamental,
Consejo Superior de Investigaciones Científicas,
Serrano 123, 28006 Madrid, SPAIN,
E-mail: mdeleon@fresno.csic.es*

³*Departamento de Matemática Fundamental, Facultad de Matemáticas,
Universidad de la Laguna, La Laguna,
Tenerife, Canary Islands, SPAIN,
E-mail: jcmarrer@ull.es, mepadron@ull.es*

February 5, 2008

Abstract

The notion of Leibniz algebroid is introduced, and it is shown that each Nambu-Poisson manifold has associated a canonical Leibniz algebroid. This fact permits to define the modular class of a Nambu-Poisson manifold as an appropriate cohomology class, extending the well-known modular class of Poisson manifolds.

Mathematics Subject Classification (1991): 53C15, 58F05, 81S10.

PACS numbers: 02.40.Ma, 03.20.+i, 0.3.65.-w

Key words and phrases: Nambu-Poisson brackets, Nambu-Poisson manifolds, Leibniz algebras, Leibniz cohomology, Leibniz algebroid, modular class.

1 Introduction

A Lie algebroid is a natural generalization of the notion of Lie algebra, and also of the tangent bundle of a manifold. There are many other interesting examples, for instance, the cotangent bundle of any Poisson manifold possesses a natural structure of Lie algebroid. Roughly speaking, a Lie algebroid over a manifold M is a vector bundle E over M such that its space of sections $\Gamma(E)$ has a structure of Lie algebra plus a mapping (the anchor map) from E onto TM which provides a Lie algebra homomorphism from $\Gamma(E)$ into the Lie algebra of vector fields $\mathfrak{X}(M)$.

The action of $\Gamma(E)$ on $C^\infty(M, \mathbb{R})$ defines the Lie algebroid cohomology of M . For a Poisson manifold M , the associated Lie algebroid is just the triple $(T^*M, \llbracket \cdot, \cdot \rrbracket, \#)$, where $\llbracket \cdot, \cdot \rrbracket$ is the bracket of 1-forms and $\#$ is the mapping from 1-forms into tangent vectors defined by the Poisson tensor. For an oriented Poisson manifold M and its associated Lie algebroid, A. Weinstein [29, 30] has defined the so-called modular class of M , which is an element of the corresponding Lie algebroid cohomology (in fact, an element of the Lichnerowicz-Poisson cohomology space $H_{LP}^1(M)$). The modular class X_ν is defined as the operator which assigns to each function f the divergence with respect to ν of its Hamiltonian vector field X_f , where ν is a volume form on M . A direct computation shows that the modular class of a symplectic manifold is null. Indeed, the vanishing of the modular class of a Poisson manifold is closely related with its regularity. Moreover, it was proved by P. Xu [31] (see also [4, 9]) that the canonical homology is dual to the Lichnerowicz-Poisson cohomology for unimodular Poisson structures, that is, for Poisson structures with null modular class. Also, it should be remarked that the modular class was the tool recently used by J.-P. Dufour and A. Haraki [8] and by Z.J. Liu and P. Xu [16] to classify quadratic Poisson structures.

Our interest is to extend the above results for Nambu-Poisson manifolds. The concept of a Nambu-Poisson structure was introduced by Takhtajan [25] in order to find an axiomatic formalism for the n -bracket operation

$$\{f_1, \dots, f_n\} = \det\left(\frac{\partial f_i}{\partial x_j}\right)$$

proposed by Nambu [24] to generalize Hamiltonian mechanics (see also [2, 5, 6, 10]). A Nambu-Poisson manifold is a manifold M endowed with a skew-symmetric n -bracket of functions $\{\cdot, \dots, \cdot\}$ satisfying the Leibniz rule and the fundamental identity

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{i=1}^n \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_i\}, \dots, g_n\},$$

for all $f_1, \dots, f_{n-1}, g_1, \dots, g_n \in C^\infty(M, \mathbb{R})$. The local and global structure of a Nambu-Poisson manifold were elucidated in recent papers [1, 11, 14, 22, 23]. Indeed, a Nambu-Poisson manifold of order greater than 2 consists of pieces which are volume manifolds,

in the same way that a Poisson manifold is made of symplectic pieces. Recently, an interesting recursive characterization of Nambu-Poisson structures was obtained in [12].

In this paper we introduce the notion of a Leibniz algebroid -a natural generalization of a Lie algebroid. The notion of Leibniz algebra was recently introduced by J.L. Loday [17, 18] (see also [19]) as a noncommutative version of Lie algebras. Indeed, a Leibniz algebra is a real vector space \mathfrak{g} endowed with a \mathbb{R} -bilinear mapping $\{ , \}$ satisfying the Leibniz identity

$$\{a_1, \{a_2, a_3\}\} - \{\{a_1, a_2\}, a_3\} - \{a_2, \{a_1, a_3\}\} = 0,$$

for all $a_1, a_2, a_3 \in \mathfrak{g}$. If the bracket is skew-symmetric we recover the notion of Lie algebra. Next, the notion of Leibniz algebroid can be introduced in the same way that for the case of Lie algebroids. One of the main results of the present paper is to associate a Leibniz algebroid to each Nambu-Poisson manifold M . The Leibniz algebroid attached to M is just the triple $(\Lambda^{n-1}(T^*M), \llbracket , \rrbracket, \#)$, where $\llbracket , \rrbracket : \Omega^{n-1}(M) \times \Omega^{n-1}(M) \longrightarrow \Omega^{n-1}(M)$ is the bracket of $(n-1)$ -forms defined by

$$\llbracket \alpha, \beta \rrbracket = \mathcal{L}_{\# \alpha} \beta + (-1)^n (i(d\alpha) \Lambda) \beta,$$

for $\alpha, \beta \in \Omega^{n-1}(M)$, and $\# : \Lambda^{n-1}(T^*M) \longrightarrow TM$ is the homomorphism of vector bundles given by $\#(\beta) = i(\beta) \Lambda$. Here Λ is the Nambu-Poisson n -vector. In addition, it is proved that the only non-null Nambu-Poisson structures of order greater than 2 on an oriented manifold M of dimension m , with $m \geq 3$, such that its Leibniz algebroid is a Lie algebroid are those defined by non-null m -vectors.

As in the case of Poisson manifolds, we define the modular class of an oriented Nambu-Poisson m -dimensional manifold M of order n . Indeed, if ν is a volume form on M then the mapping

$$\mathcal{M}_\nu : C^\infty(M, \mathbb{R}) \times \dots^{(n-1)} \dots \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R}),$$

given by

$$\mathcal{L}_{X_{f_1 \dots f_{n-1}}} \nu = \mathcal{M}_\nu(f_1, \dots, f_{n-1}) \nu$$

is a $(n-1)$ -vector on M , where $X_{f_1 \dots f_{n-1}} = \#(df_1 \wedge \dots \wedge df_{n-1})$ is the Hamiltonian vector field associated with the functions f_1, \dots, f_{n-1} . Next, the mapping

$$\mathcal{M}_\Lambda : \Omega^{n-1}(M) \longrightarrow C^\infty(M, \mathbb{R}) \quad \alpha \mapsto i(\alpha) \mathcal{M}_\nu$$

defines a 1-cocycle in the Leibniz cohomology complex associated to the Leibniz algebroid $(\Lambda^{n-1}(T^*M), \llbracket , \rrbracket, \#)$. The cohomology class $[\mathcal{M}_\Lambda] \in H^1(\Omega^{n-1}(M); C^\infty(M, \mathbb{R}))$ does not depend on the chosen volume form and it is called the modular class of M . As in the case of Poisson manifolds it is proved that the modular class of a volume manifold is null. In a forthcoming paper we will investigate the role played for the modular class in the problem of classification of Nambu-Poisson manifolds. We also are investigating the existence of a dual homology to the Leibniz algebroid cohomology, which would be related with the vanishing of the modular class (see [4, 9, 31] for the case of Lie algebroids and Poisson manifolds).

2 Preliminaries

All the manifolds considered in this paper are assumed to be connected.

2.1 Nambu-Poisson structures

Let M be a differentiable manifold of dimension m . Denote by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M , by $C^\infty(M, \mathbb{R})$ the algebra of C^∞ real-valued functions on M and by $\Omega^k(M)$ the space of k -forms on M .

An *almost Poisson bracket* of order n ($n \leq m$) on M (see [14]) is an n -linear mapping $\{\cdot, \dots, \cdot\} : C^\infty(M, \mathbb{R}) \times \dots \times C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R})$ satisfying the following properties :

(1) (*Skew-symmetry*)

$$\{f_1, \dots, f_n\} = (-1)^{\varepsilon(\sigma)} \{f_{\sigma(1)}, \dots, f_{\sigma(n)}\}, \quad (2.1)$$

for all $f_1, \dots, f_n \in C^\infty(M, \mathbb{R})$ and $\sigma \in \text{Symm}(n)$, where $\text{Symm}(n)$ is a symmetric group of n elements and $\varepsilon(\sigma)$ is the parity of the permutation σ .

(2) (*Leibniz rule*)

$$\{f_1 g_1, f_2, \dots, f_n\} = f_1 \{g_1, f_2, \dots, f_n\} + g_1 \{f_1, f_2, \dots, f_n\}, \quad (2.2)$$

for all $f_1, \dots, f_n, g_1 \in C^\infty(M, \mathbb{R})$.

If $\{\cdot, \dots, \cdot\}$ is an almost Poisson bracket of order n then we define a skew-symmetric tensor Λ of type $(n, 0)$ (n -vector) as follows

$$\Lambda(df_1, \dots, df_n) = \{f_1, \dots, f_n\},$$

for $f_1, \dots, f_n \in C^\infty(M, \mathbb{R})$. Conversely, given an n -vector on M , the above formula defines an almost Poisson bracket of order n . The pair (M, Λ) is called a *generalized almost Poisson manifold of order n* .

If $\Lambda^{n-1}(T^*M)$ is the vector bundle of the $(n-1)$ -forms on M then Λ induces a homomorphism of vector bundles

$$\# : \Lambda^{n-1}(T^*M) \longrightarrow TM$$

by defining

$$\#(\beta) = i(\beta)\Lambda(x) \quad (2.3)$$

for $\beta \in \Lambda^{n-1}(T_x^*M)$ and $x \in M$, where $i(\beta)$ is the contraction by β . Denote also by

$$\# : \Omega^{n-1}(M) \rightarrow \mathfrak{X}(M)$$

the homomorphism of $C^\infty(M, \mathbb{R})$ -modules given by

$$\#(\alpha)(x) = \#(\alpha(x)) \quad (2.4)$$

for all $\alpha \in \Omega^{n-1}(M)$ and $x \in M$. Then, if f_1, \dots, f_{n-1} are $n-1$ functions on M , we define a vector field

$$X_{f_1 \dots f_{n-1}} = \#(df_1 \wedge \dots \wedge df_{n-1}), \quad (2.5)$$

which is called the *Hamiltonian vector field* associated with the Hamiltonian functions f_1, \dots, f_{n-1} . From (2.5) it follows that

$$X_{f_1 \dots f_{n-1}}(f_n) = \{f_1, \dots, f_{n-1}, f_n\}. \quad (2.6)$$

A more rich structure, related with interesting dynamical problems, can be considered adding to the almost Poisson bracket $\{\dots, \}$ the following integrability condition (*fundamental identity*)

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{i=1}^n \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_i\}, \dots, g_n\} \quad (2.7)$$

for all $f_1, \dots, f_{n-1}, g_1, \dots, g_n$ functions on M . In this case, $\{\dots, \}$ is called a *Nambu-Poisson bracket* and (M, Λ) is a *Nambu-Poisson manifold of order n* (see [25]).

In fact, an n -vector Λ on M defines a Nambu-Poisson structure if the Hamiltonian vector fields are derivations of the algebra $(C^\infty(M, \mathbb{R}) \times \dots \times C^\infty(M, \mathbb{R}), \{\dots, \})$ or equivalently, every Hamiltonian vector field $X_{f_1 \dots f_{n-1}}$ is an infinitesimal automorphism of Λ , that is,

$$\mathcal{L}_{X_{f_1 \dots f_{n-1}}} \Lambda = 0.$$

Examples 2.1 (i) The *Poisson manifolds* are just the Nambu-Poisson manifolds of order 2 [15, 26, 28].

Another examples of Nambu-Poisson manifolds are the following.

(ii) Let N be an oriented m -dimensional manifold and choose a volume form ν_N on N . Given m functions f_1, \dots, f_m on N , we define its m -bracket by the formula

$$df_1 \wedge \dots \wedge df_m = \{f_1, \dots, f_m\} \nu_N. \quad (2.8)$$

It is not hard to prove that it is a Nambu-Poisson bracket (see [11]). Denote by Λ_{ν_N} the m -vector associated with this bracket. Note that for the Nambu-Poisson structure Λ_{ν_N} the homomorphism $\# : \Omega^{m-1}(N) \longrightarrow \mathfrak{X}(N)$ is an isomorphism. Furthermore, if Λ is a Nambu-Poisson structure of order m and $\Lambda \neq 0$ at every point then there exists a volume form ν on N such that $\Lambda = \Lambda_{\nu_N}$ (see [14]).

(iii) Let Λ_N be an arbitrary m -vector on an oriented m -dimensional manifold N with volume form ν_N . Then, there exists a function $f \in C^\infty(N, \mathbb{R})$ such that $\Lambda_N = f \Lambda_{\nu_N}$.

Moreover, if f_1, \dots, f_{m-1} are $m - 1$ functions on N and $X_{f_1 \dots f_{m-1}}$ is the corresponding Hamiltonian vector field with respect to the m -vector Λ_N , it follows that $\mathcal{L}_{X_{f_1 \dots f_{m-1}}} \Lambda_N = 0$. Thus, we deduce that (N, Λ_N) is a Nambu-Poisson manifold of order m .

(iv) If V is a k -dimensional differentiable manifold, Λ_N induces an m -vector Λ on the product $N \times V$ and $(N \times V, \Lambda)$ is a Nambu-Poisson manifold of order m .

The following theorem describes the local structure of the Nambu-Poisson brackets of order n , with $n \geq 3$.

Theorem 2.2 [1, 11, 14, 22, 23] *Let M be a differentiable manifold of dimension m . The n -vector Λ , $n \geq 3$, defines a Nambu-Poisson structure on M if and only if for all $x \in M$ where $\Lambda(x) \neq 0$, there exist local coordinates $(x^1, \dots, x^n, x^{n+1}, \dots, x^m)$ around x such that*

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^n}.$$

Remark 2.3 A point x of a Nambu-Poisson manifold (M, Λ) of order $n \geq 3$ is said to be *regular* if $\Lambda(x) \neq 0$.

Remark 2.4 Let (M, Λ) be a Nambu-Poisson manifold of order n , with $n \geq 3$, and consider the *characteristic distribution* \mathcal{D} on M given by

$$x \in M \longrightarrow \mathcal{D}(x) = \#(\Lambda^{n-1} T_x^* M) = \langle \{X_{f_1 \dots f_{n-1}}(x)/f_1, \dots, f_{n-1} \in C^\infty(M, \mathbb{R})\} \rangle \subseteq T_x M.$$

Then, \mathcal{D} defines a generalization foliation on M whose leaves are either points or n -dimensional manifolds endowed with a Nambu-Poisson structure coming from a volume form (see [14]).

2.2 Leibniz algebras and cohomology

First, we recall the definition of real Leibniz algebra (see [17, 18, 19]).

A *Leibniz algebra structure* on a real vector space \mathfrak{g} is a \mathbb{R} -bilinear map $\{ , \} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the *Leibniz identity*, that is,

$$\{a_1, \{a_2, a_3\}\} - \{\{a_1, a_2\}, a_3\} - \{a_2, \{a_1, a_3\}\} = 0, \quad (2.9)$$

for $a_1, a_2, a_3 \in \mathfrak{g}$. In such a case, one says that $(\mathfrak{g}, \{ , \})$ is a *Leibniz algebra*.

Moreover, if the skew-symmetric condition is required then $(\mathfrak{g}, \{ , \})$ is a Lie algebra. In this sense, a Leibniz algebra is a non-commutative version of a Lie algebra.

Let $(\mathfrak{g}, \{ , \})$ be a Leibniz algebra and \mathcal{M} be a real vector space endowed with a \mathbb{R} -bilinear map

$$\mathfrak{g} \times \mathcal{M} \longrightarrow \mathcal{M}$$

such that $\{a_1, a_2\}m = a_1(a_2m) - a_2(a_1m)$, for all $a_1, a_2 \in \mathfrak{g}$ and $m \in \mathcal{M}$. Then \mathcal{M} is a \mathfrak{g} -module relative to the representation of \mathfrak{g} on \mathcal{M} .

If \mathcal{M} is a \mathfrak{g} -module we can introduce a cohomology complex as follows.

A k -linear mapping $c^k : \mathfrak{g} \times \dots^{(k)} \dots \times \mathfrak{g} \longrightarrow \mathcal{M}$ is called a \mathcal{M} -valued k -cochain. We denote by $C^k(\mathfrak{g}; \mathcal{M})$ the real vector space of these cochains.

The operator $\partial^k : C^k(\mathfrak{g}; \mathcal{M}) \longrightarrow C^{k+1}(\mathfrak{g}; \mathcal{M})$ given by

$$\begin{aligned} \partial^k c^k(a_0, \dots, a_k) &= \sum_{i=0}^k (-1)^i a_i c^k(a_0, \dots, \widehat{a_i}, \dots, a_k) \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i-1} c^k(a_0, \dots, \widehat{a_i}, \dots, a_{j-1}, \{a_i, a_j\}, a_{j+1}, \dots, a_k), \end{aligned} \quad (2.10)$$

defines a coboundary since $\partial^{k+1} \circ \partial^k = 0$. Hence, $(C^*(\mathfrak{g}; \mathcal{M}), \partial)$ is a cohomology complex and the corresponding cohomology spaces

$$H^k(\mathfrak{g}; \mathcal{M}) = \frac{\ker\{\partial^k : C^k(\mathfrak{g}; \mathcal{M}) \rightarrow C^{k+1}(\mathfrak{g}; \mathcal{M})\}}{\text{Im}\{\partial^{k-1} : C^{k-1}(\mathfrak{g}; \mathcal{M}) \rightarrow C^k(\mathfrak{g}; \mathcal{M})\}},$$

are called the *Leibniz cohomology groups of \mathfrak{g} with coefficients in \mathcal{M}* (see [17, 18, 19]).

Note that if $(\mathfrak{g}, \{ , \})$ is a Lie algebra and c^k is a skew-symmetric \mathcal{M} -valued k -cochain then $\partial^k c^k$ is a skew-symmetric \mathcal{M} -valued $(k+1)$ -cochain. Thus, we can consider the subcomplex $(C_{Lie}^*(\mathfrak{g}; \mathcal{M}), \partial)$ of $(C^*(\mathfrak{g}; \mathcal{M}), \partial)$ that consists of the skew-symmetric \mathcal{M} -valued cochains. In fact, the cohomology of this subcomplex is just the cohomology $H_{Lie}^*(\mathfrak{g}; \mathcal{M})$ of the Lie algebra \mathfrak{g} with coefficients in \mathcal{M} . Therefore, we have defined a natural homomorphism

$$i^k : H_{Lie}^k(\mathfrak{g}; \mathcal{M}) \rightarrow H^k(\mathfrak{g}; \mathcal{M})$$

between the cohomology groups $H_{Lie}^k(\mathfrak{g}; \mathcal{M})$ and $H^k(\mathfrak{g}; \mathcal{M})$.

Examples 2.5 (i) Let M be a differentiable manifold and $(\mathfrak{X}(M), [,])$ the Lie algebra of the vector fields on M . Then, the real vector space $C^\infty(M, \mathbb{R})$ is a $\mathfrak{X}(M)$ -module with the usual multiplication

$$\mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R}), \quad (X, f) \mapsto X(f).$$

The k -cochains in the Leibniz cohomology complex are the k -linear mappings $c^k : \mathfrak{X}(M) \times \dots^{(k)} \dots \times \mathfrak{X}(M) \rightarrow C^\infty(M, \mathbb{R})$ and the Leibniz cohomology operator $d : C_{Leib}^k(M) =$

$C^k(\mathfrak{X}(M); C^\infty(M, \mathbb{R})) \longrightarrow C_{Leib}^{k+1}(M) = C^{k+1}(\mathfrak{X}(M); C^\infty(M, \mathbb{R}))$ is defined as the exterior differential operator, that is,

$$\begin{aligned} dc^k(X_0, \dots, X_k) &= \sum_{i=0}^k (-1)^i X_i(c^k(X_0, \dots, \widehat{X}_i, \dots, X_k)) \\ &\quad + \sum_{0 \leq i < j \leq k} (-1)^{i-1} c^k(X_0, \dots, \widehat{X}_i, \dots, X_{j-1}, [X_i, X_j], X_{j+1}, \dots, X_k), \end{aligned} \quad (2.11)$$

for all $X_0, \dots, X_k \in \mathfrak{X}(M)$.

The resultant cohomology $H^*(\mathfrak{X}(M); C^\infty(M, \mathbb{R}))$ is the *Leibniz cohomology* of M and it is denoted by $H_{Leib}^*(M)$ (for a detailed study of this cohomology, we refer to [20]). Note that the de Rham cohomology of M , $H_{dR}^*(M)$, is just the cohomology of the subcomplex of the skew-symmetric $C^\infty(M, \mathbb{R})$ -valued cochains that are $C^\infty(M, \mathbb{R})$ -linear.

(ii) Let (M, Λ) be a Nambu-Poisson manifold of order n with Nambu-Poisson bracket $\{, \dots, \}$. Consider on $\Lambda^{n-1}(C^\infty(M, \mathbb{R}))$ the bracket $\{, \dots, \}'$ characterized by the formula:

$$\{f_1 \wedge \dots \wedge f_{n-1}, g_1 \wedge \dots \wedge g_{n-1}\}' = \sum_{i=1}^{n-1} g_1 \wedge \dots \wedge \{f_1, \dots, f_{n-1}, g_i\} \wedge \dots \wedge g_{n-1} \quad (2.12)$$

for $f_1, \dots, f_{n-1}, g_1, \dots, g_{n-1} \in C^\infty(M, \mathbb{R})$. Using (2.7), we deduce that $(\Lambda^{n-1}(C^\infty(M, \mathbb{R})), \{, \dots, \}')$ is a Leibniz algebra (see [7, 11, 23]). Moreover, the real vector space $C^\infty(M, \mathbb{R})$ is a $\Lambda^{n-1}(C^\infty(M, \mathbb{R}))$ -module with the multiplication

$$\Lambda^{n-1}(C^\infty(M, \mathbb{R})) \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$$

characterized by

$$(f_1 \wedge \dots \wedge f_{n-1}, f) \mapsto X_{f_1 \dots f_{n-1}}(f). \quad (2.13)$$

The resultant cohomology

$$H^*(\Lambda^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R}))$$

was studied in [7] and [11].

3 Leibniz algebroids and Nambu-Poisson manifolds

In this section we will define a generalization of the notion of Lie algebroid (the Leibniz algebroid) and we will prove that a Nambu-Poisson manifold has associated a structure of this type.

Definition 3.1 *A Leibniz algebroid structure on a differentiable vector bundle $\pi : E \rightarrow M$ is a pair that consists of a Leibniz algebra structure \llbracket, \rrbracket on the space $\Gamma(E)$ of the global cross sections of $\pi : E \rightarrow M$ and a vector bundle morphism $\varrho : E \rightarrow TM$, called the anchor map, such that the induced map $\varrho : \Gamma(E) \rightarrow \Gamma(TM) = \mathfrak{X}(M)$ satisfies the following relations:*

$$(i) \quad \varrho \llbracket s_1, s_2 \rrbracket = [\varrho(s_1), \varrho(s_2)],$$

$$(ii) \quad \llbracket s_1, f s_2 \rrbracket = f \llbracket s_1, s_2 \rrbracket + \varrho(s_1)(f) s_2,$$

for all $s_1, s_2 \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$.

A triple $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$ is called a *Leibniz algebroid* over M .

Remark 3.2 (i) If $(\mathfrak{g}, \{ \cdot, \cdot \})$ is a Leibniz algebra then $(\mathfrak{g}, \{ \cdot, \cdot \}, \varrho \equiv 0)$ is a Leibniz algebroid over a point.

(ii) Every Lie algebroid over a manifold M is trivially a Leibniz algebroid. In fact, a Leibniz algebroid $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$ over M is a Lie algebroid if and only if the Leibniz bracket $\llbracket \cdot, \cdot \rrbracket$ on $\Gamma(E)$ is skew-symmetric.

If (M, Λ) is a Poisson manifold then it is possible to define a Lie algebra structure $\llbracket \cdot, \cdot \rrbracket$ on the space of 1-forms $\Omega^1(M)$ in such a sense that the triple $(T^*M, \llbracket \cdot, \cdot \rrbracket, \#)$ is a Lie algebroid over M , where T^*M is the cotangent bundle of M and $\# : T^*M \rightarrow TM$ is the homomorphism of vector bundles given by (2.4) (see [3, 26]).

Next, we will prove that associated to a Nambu-Poisson manifold of order n , with $n \geq 3$, there is a canonical Leibniz algebroid.

Let (M, Λ) be an m -dimensional Nambu-Poisson manifold of order $n, n \geq 3$, with Nambu-Poisson bracket $\{ \cdot, \dots, \cdot \}$.

Proposition 3.3 For all $\alpha, \beta \in \Omega^{n-1}(M)$ we have

$$[\# \alpha, \# \beta] = \#(\mathcal{L}_{\# \alpha} \beta + (-1)^n (i(d\alpha) \Lambda) \beta)$$

where $\# : \Omega^{n-1}(M) \rightarrow \mathfrak{X}(M)$ is the homomorphism defined in (2.3) and (2.4) and \mathcal{L} is the Lie derivative operator.

Proof: Using (2.3) and (2.4) we have that

$$\begin{aligned} [\# \alpha, \# \beta] &= i(\beta)(\mathcal{L}_{\# \alpha} \Lambda) + i(\mathcal{L}_{\# \alpha} \beta) \Lambda \\ &= i(\beta)(\mathcal{L}_{\# \alpha} \Lambda) + \#(\mathcal{L}_{\# \alpha} \beta). \end{aligned} \tag{3.1}$$

On the other hand,

$$\mathcal{L}_{\# \alpha} \Lambda = (-1)^n (i(d\alpha) \Lambda) \Lambda. \tag{3.2}$$

Indeed, if $x \in M$ and $\Lambda(x) = 0$ then $(\mathcal{L}_{\# \alpha} \Lambda)(x) = (-1)^n (i(d\alpha) \Lambda)(x) \Lambda(x) = 0$.

If $x \in M$ and $\Lambda(x) \neq 0$, then (see Theorem 2.2) there exist local coordinates $(x^1, \dots, x^n, x^{n+1}, \dots, x^m)$ in an open subset U of M , $x \in U$, such that

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^n}. \tag{3.3}$$

So, to prove (3.2) it suffices of course to check this formula for local $(n - 1)$ -forms

$$\alpha = \sum_{i=1}^n \alpha_i dx_1 \wedge \dots \widehat{dx_i} \wedge \dots \wedge dx_n,$$

with $\alpha_i \in C^\infty(U, \mathbb{R})$. Now, from (2.3), (2.4) and (3.3), one deduces that (3.2) is true for this type of $(n - 1)$ -forms.

Finally, using (2.3), (2.4), (3.1) and (3.2), the result follows. \square

The above result suggests us to introduce the following definition.

Definition 3.4 *Let (M, Λ) be an m -dimensional Nambu-Poisson manifold of order n , with $3 \leq n \leq m$. The bracket of $(n - 1)$ -forms on M is the \mathbb{R} -bilinear operation $\llbracket \cdot, \cdot \rrbracket : \Omega^{n-1}(M) \times \Omega^{n-1}(M) \longrightarrow \Omega^{n-1}(M)$ given by*

$$\llbracket \alpha, \beta \rrbracket = \mathcal{L}_{\# \alpha} \beta + (-1)^n (i(d\alpha) \Lambda) \beta, \quad (3.4)$$

for $\alpha, \beta \in \Omega^{n-1}(M)$.

The mapping $\llbracket \cdot, \cdot \rrbracket$ is characterized as follows:

Theorem 3.5 *Let (M, Λ) be an m -dimensional Nambu-Poisson manifold of order n , with $3 \leq n \leq m$. Then exists a unique \mathbb{R} -bilinear operation $\llbracket \cdot, \cdot \rrbracket : \Omega^{n-1}(M) \times \Omega^{n-1}(M) \longrightarrow \Omega^{n-1}(M)$ such that:*

(i) *For all $f_1, \dots, f_{n-1}, g_1, \dots, g_{n-1} \in C^\infty(M, \mathbb{R})$, we have*

$$\llbracket df_1 \wedge \dots \wedge df_{n-1}, dg_1 \wedge \dots \wedge dg_{n-1} \rrbracket = \sum_{i=1}^{n-1} dg_1 \wedge \dots \wedge d\{f_1, \dots, f_{n-1}, g_i\} \wedge \dots \wedge dg_{n-1}. \quad (3.5)$$

(ii) *For all $f \in C^\infty(M, \mathbb{R})$ and $\alpha, \beta \in \Omega^{n-1}(M)$, we have*

$$\llbracket \alpha, f\beta \rrbracket = f\llbracket \alpha, \beta \rrbracket + \# \alpha(f) \beta, \quad (3.6)$$

$$\llbracket f\alpha, \beta \rrbracket = f\llbracket \alpha, \beta \rrbracket - i(\# \alpha)(df \wedge \beta). \quad (3.7)$$

This operation is given by (3.4).

Proof: It is easy to prove that the bracket defined in (3.4) satisfies (3.5), (3.6) and (3.7).

Now, suppose that $\llbracket \cdot, \cdot \rrbracket_1 : \Omega^{n-1}(M) \times \Omega^{n-1}(M) \longrightarrow \Omega^{n-1}(M)$ is a \mathbb{R} -bilinear operation which satisfies (3.5), (3.6) and (3.7). Then $\llbracket \cdot, \cdot \rrbracket_1$ must be of the local type, i.e., $\llbracket \alpha, \beta \rrbracket_1(x_0)$ will depend on α and β around x_0 only, for all $x_0 \in M$. Indeed, if $\beta_{1|U} = \beta_{2|U}$ for an open neighborhood U of x_0 , and if f is a C^∞ real-valued function that vanishes outside U , and

equals 1 on a compact neighborhood $V_{x_0} \subseteq U$, then $\sigma = f\beta_1 = f\beta_2$ is well defined on M and, by (3.6) we have

$$\llbracket \alpha, \sigma \rrbracket_1(x_0) = \llbracket \alpha, \beta_1 \rrbracket_1(x_0), \quad \llbracket \alpha, \sigma \rrbracket_1(x_0) = \llbracket \alpha, \beta_2 \rrbracket_1(x_0),$$

i.e., $\llbracket \alpha, \beta_1 \rrbracket_1(x_0) = \llbracket \alpha, \beta_2 \rrbracket_1(x_0)$.

Similarly, if $\alpha_1|_U = \alpha_2|_U$ and $\nu = f\alpha_1 = f\alpha_2$ then, from (3.7), we deduce that

$$\llbracket \nu, \beta \rrbracket_1(x_0) = \llbracket \alpha_1, \beta \rrbracket_1(x_0), \quad \llbracket \nu, \beta \rrbracket_1(x_0) = \llbracket \alpha_2, \beta \rrbracket_1(x_0),$$

that is, $\llbracket \alpha_1, \beta \rrbracket_1(x_0) = \llbracket \alpha_2, \beta \rrbracket_1(x_0)$.

Next, we will show that $\llbracket \cdot, \cdot \rrbracket_1 = \llbracket \cdot, \cdot \rrbracket$.

Let x be a point of M and α and β $(n-1)$ -forms on M .

Assume that (x^1, \dots, x^m) are local coordinates in an open neighborhood U of x and that in U we have

$$\begin{aligned} \alpha &= \sum_{1 \leq i_1 < \dots < i_{n-1} \leq m} \alpha_{i_1 \dots i_{n-1}} dx^{i_1} \wedge \dots \wedge dx^{i_{n-1}} \\ \beta &= \sum_{1 \leq j_1 < \dots < j_{n-1} \leq m} \beta_{j_1 \dots j_{n-1}} dx^{j_1} \wedge \dots \wedge dx^{j_{n-1}}. \end{aligned}$$

Using (2.5), (3.4), (3.5), (3.6), (3.7) and the local character of the bracket $\llbracket \cdot, \cdot \rrbracket_1$, we obtain that

$$\begin{aligned} \llbracket \alpha, \beta \rrbracket_1(x) &= \sum_{\substack{1 \leq i_1 < \dots < i_{n-1} \leq m \\ 1 \leq j_1 < \dots < j_{n-1} \leq m}} \left[\sum_{k=1, \dots, n-1} (\alpha_{i_1 \dots i_{n-1}} \beta_{j_1 \dots j_{n-1}} dx^{j_1} \wedge \dots \wedge dx^{j_{k-1}} \wedge \right. \\ &\quad \wedge d\{x^{i_1}, \dots, x^{i_{n-1}}, x^{j_k}\} \wedge dx^{j_{k+1}} \wedge \dots \wedge dx^{j_{n-1}}) \\ &\quad - \beta_{j_1 \dots j_{n-1}} X_{x^{i_1} \dots x^{i_{n-1}}}(\alpha_{i_1 \dots i_{n-1}}) dx^{j_1} \wedge \dots \wedge dx^{j_{n-1}} \\ &\quad + \beta_{j_1 \dots j_{n-1}} d\alpha_{i_1 \dots i_{n-1}} \wedge i_{X_{x^{i_1} \dots x^{i_{n-1}}}}(dx^{j_1} \wedge \dots \wedge dx^{j_{n-1}}) \\ &\quad \left. + \alpha_{i_1 \dots i_{n-1}} X_{x^{i_1} \dots x^{i_{n-1}}}(\beta_{j_1 \dots j_{n-1}}) dx^{j_1} \wedge \dots \wedge dx^{j_{n-1}} \right] (x) \\ &= \llbracket \alpha, \beta \rrbracket(x). \end{aligned}$$

From the arbitrariness of the point x , it follows that $\llbracket \alpha, \beta \rrbracket_1 = \llbracket \alpha, \beta \rrbracket$. \square

Now, we will prove that a Nambu-Poisson manifold of order n , with $n \geq 3$, has associated a Leibniz algebroid.

Theorem 3.6 *Let (M, Λ) be an m -dimensional Nambu-Poisson manifold of order n , with $3 \leq n \leq m$. Then, the triple $(\Lambda^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$ is a Leibniz algebroid over M , where $\llbracket \cdot, \cdot \rrbracket : \Omega^{n-1}(M) \times \Omega^{n-1}(M) \longrightarrow \Omega^{n-1}(M)$ is the bracket of $(n-1)$ -forms defined by (3.4) and $\# : \Lambda^{n-1}(T^*M) \longrightarrow TM$ is the homomorphism of vector bundles given by (2.3).*

Proof: We must prove that

$$\llbracket \alpha, \llbracket \beta, \gamma \rrbracket \rrbracket - \llbracket \llbracket \alpha, \beta \rrbracket, \gamma \rrbracket - \llbracket \beta, \llbracket \alpha, \gamma \rrbracket \rrbracket = 0, \quad (3.8)$$

for $\alpha, \beta, \gamma \in \Omega^{n-1}(M)$.

From (3.2), (3.4) and Proposition 3.3, we obtain that

$$(i(d[\alpha, \beta])\Lambda - \# \alpha(i(d\beta)\Lambda) + \# \beta(i(d\alpha)\Lambda))\Lambda = 0.$$

Thus,

$$i(d[\alpha, \beta])\Lambda = \# \alpha(i(d\beta)\Lambda) - \# \beta(i(d\alpha)\Lambda). \quad (3.9)$$

On the other hand, using (3.4), we have that

$$[\alpha, [\beta, \gamma]] - [\beta, [\alpha, \gamma]] = \mathcal{L}_{[\# \alpha, \# \beta] \gamma} + (-1)^n (\# \alpha(i(d\beta)\Lambda) - \# \beta(i(d\alpha)\Lambda))\gamma,$$

which implies that (see (3.9) and Proposition 3.3)

$$[\alpha, [\beta, \gamma]] - [\beta, [\alpha, \gamma]] = \mathcal{L}_{\# [\alpha, \beta]} \gamma + (-1)^n (i(d[\alpha, \beta])\Lambda)\gamma. \quad (3.10)$$

Therefore, from (3.4) and (3.10), it follows that (3.8) holds. Hence, we deduce that the bracket $[\ , \]$ induces a Leibniz algebra structure on $\Omega^{n-1}(M)$.

Using this fact, (3.6) and Proposition 3.3, we conclude that the triple $(\wedge^{n-1}(T^*M), [\ , \], \#)$ is a Leibniz algebroid over M . \square

Remark 3.7 In [13] the authors have introduced the notion of Filippov algebroid, as a n -ary generalization of Lie algebroids. Indeed, the binary bracket of sections in a Lie algebroid is replaced in a Filippov algebroid $E \rightarrow M$ by an n -bracket $[\ , \dots, \]$ on $\Gamma(E)$ satisfying the fundamental identity and, the anchor map is a vector bundle morphism $\wedge^{n-1}(E) \rightarrow TM$ compatible with the n -bracket. In [13, 27] an n -bracket of 1-forms on a Nambu-Poisson manifold is defined. However, this bracket does not satisfy the fundamental identity.

In general, the bracket defined in (3.4) is not skew-symmetric and consequently the Leibniz algebroid $(\wedge^{n-1}(T^*M), [\ , \], \#)$ is not a Lie algebroid.

In the following result we characterize when this Leibniz algebroid is a Lie algebroid on an oriented manifold.

Theorem 3.8 *Let M be an oriented manifold of dimension m , $m \geq 3$. The unique non-null Nambu-Poisson structures of order greater than 2 on M such that the Leibniz algebroid is a Lie algebroid are those defined by non-null m -vectors.*

Proof: Suppose that Λ is a non-null m -vector. Then (M, Λ) is a Nambu-Poisson manifold of order m (see Examples 2.1).

Now, if α and β are $(m-1)$ -forms on M , we consider the $(m-1)$ -form σ on M defined by

$$\sigma = [\alpha, \beta] + [\beta, \alpha].$$

We must prove that $\sigma = 0$.

Since the set

$$A = \{x \in M / \Lambda(x) \neq 0\}$$

is an open subset of M , Λ induces a Nambu-Poisson structure Λ_A on A of order m which is non-null at every point. Then, as M is oriented, we deduce that Λ_A is defined by a volume form on A and the corresponding homomorphism

$$\#_A : \Omega^{n-1}(A) \longrightarrow \mathfrak{X}(A),$$

given by (2.3) and (2.4), is an isomorphism. Using this last fact, Proposition 3.3 and the skew-symmetry of the Lie bracket $[\cdot, \cdot]$ of vector fields, we obtain that $\sigma = 0$ on A .

On the other hand, it is obvious that σ is null on the exterior of A (note that the exterior of A is an open subset of M and that $\Lambda = 0$ on such a set). Finally, by continuity we conclude that $\sigma = 0$ on the boundary of A . Thus, $\sigma = 0$ on M .

Conversely, suppose that (M, Λ) is an oriented m -dimensional Nambu-Poisson manifold of order n , with $3 \leq n \leq m$ and that $(\Lambda^{n-1}(T^*M), [\cdot, \cdot], \#)$ is a Lie algebroid.

Since Λ is a non-null n -vector, there exists a point of M such that $\Lambda(x) \neq 0$ and there exist local coordinates $(x^1, \dots, x^n, x^{n+1}, \dots, x^m)$ on an open neighborhood U of x such that the n -vector Λ_U induced by Λ on U is given by (see Theorem 2.2)

$$\Lambda_U = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^n}.$$

Using the fact that $(\Lambda^{n-1}(T^*M), [\cdot, \cdot], \#)$ is a Lie algebroid, we deduce that the bracket $[\cdot, \cdot]_U : \Omega^{n-1}(U) \times \Omega^{n-1}(U) \longrightarrow \Omega^{n-1}(U)$ defined by Λ_U (see (3.4)) is skew-symmetric.

Now, if $n < m$ we can consider the $n - 1$ forms on U given by

$$\begin{aligned} \alpha &= dx^1 \wedge \dots \wedge dx^{n-3} \wedge dx^n \wedge dx^{n+1}, \\ \beta &= x^{n-2} dx^1 \wedge \dots \wedge dx^{n-1}, \end{aligned}$$

and a direct computation proves that $0 = [\alpha, \beta]_U \neq -[\beta, \alpha]_U$. This is a contradiction. Hence, $n = m$. \square

From Theorem 3.8, we obtain:

Corollary 3.9 *Let M be an oriented manifold of dimension m , $m \geq 3$, and ν be a volume form on M . Then the Leibniz algebroid associated with the Nambu-Poisson manifold (M, Λ_ν) is a Lie algebroid.*

4 Cohomology of a Leibniz algebroid and modular class of a Nambu-Poisson manifold

Let $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$ be a Leibniz algebroid over a manifold M . From Definition 3.1, we deduce that $C^\infty(M, \mathbb{R})$ is a $\Gamma(E)$ -module with the multiplication

$$\Gamma(E) \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R}) \quad (s, f) \mapsto \varrho(s)(f). \quad (4.1)$$

Thus, we can consider the differential complex $(C^*(\Gamma(E); C^\infty(M, \mathbb{R})), \partial)$ and its cohomology $H^*(\Gamma(E); C^\infty(M, \mathbb{R}))$, that is, the cohomology of $\Gamma(E)$ with coefficients in $C^\infty(M, \mathbb{R})$ (see Section 2.2). $H^*(\Gamma(E); C^\infty(M, \mathbb{R}))$ is called *the Leibniz algebroid cohomology of E* . Using (2.10), we have that

$$\begin{aligned} \partial^k c^k(s_0, \dots, s_k) &= \sum_{i=0}^k (-1)^i \varrho(s_i)(c^k(s_0, \dots, \widehat{s}_i, \dots, s_k)) \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i-1} c^k(s_0, \dots, \widehat{s}_i, \dots, s_{j-1}, \llbracket s_i, s_j \rrbracket, s_{j+1}, \dots, s_k) \end{aligned} \quad (4.2)$$

for $c^k \in C^k(\Gamma(E); C^\infty(M, \mathbb{R}))$ and $s_0, \dots, s_k \in \Gamma(E)$.

Remark 4.1 (i) Let $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$ be a Lie algebroid over M and $c^k \in C^k(\Gamma(E); C^\infty(M, \mathbb{R}))$. If c^k is skew-symmetric and $C^\infty(M, \mathbb{R})$ -linear then $\partial^k c^k$ is also skew-symmetric and $C^\infty(M, \mathbb{R})$ -linear.

The *Lie algebroid cohomology* of E is the cohomology of the subcomplex of the cochains which are skew-symmetric and $C^\infty(M, \mathbb{R})$ -linear (see [21]).

(ii) If $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$ is a Leibniz algebroid over M and $c^k \in C^k(\Gamma(E); C^\infty(M, \mathbb{R}))$ is skew-symmetric (respectively, $C^\infty(M, \mathbb{R})$ -linear) then, in general, $\partial^k c^k$ is not skew-symmetric (respectively, $C^\infty(M, \mathbb{R})$ -linear).

The following result relates the Leibniz algebroid cohomology of E with the Leibniz cohomology of the base manifold M .

Proposition 4.2 *Let $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$ be a Leibniz algebroid over a manifold M . Suppose that $(C_{Leib}^*(M), d)$ is the Leibniz cohomology complex of the manifold M and denote by*

$$\tilde{\varrho}^k : C_{Leib}^k(M) \longrightarrow C^k(\Gamma(E); C^\infty(M, \mathbb{R}))$$

the homomorphism defined by

$$\tilde{\varrho}^k(c^k)(s_1, \dots, s_k) = c^k(\varrho(s_1), \dots, \varrho(s_k))$$

for $c^k \in C_{Leib}^k(M)$ and $s_1, \dots, s_k \in \Gamma(E)$. Then, the mappings $\tilde{\varrho}^k$ induce a homomorphism of complexes

$$\tilde{\varrho} : (C_{Leib}^*(M), d) \longrightarrow (C^*(\Gamma(E); C^\infty(M, \mathbb{R})), \partial).$$

Therefore, we have the corresponding homomorphism in cohomology

$$\tilde{\varrho} : H_{Leib}^*(M) \longrightarrow H^*(\Gamma(E); C^\infty(M, \mathbb{R})).$$

Proof: It follows using (2.11), (4.2) and Definition 3.1. \square

Remark 4.3 In fact, if $(E, \llbracket \cdot, \cdot \rrbracket, \varrho)$ is a Lie algebroid over M , we can define a homomorphism $\bar{\varrho}$ between the de Rham cohomology of M , $H_{dR}^*(M)$, and the Leibniz algebroid cohomology of E given by

$$\bar{\varrho} = \tilde{\varrho} \circ i : H_{dR}^*(M) \longrightarrow H_{Leib}^*(M) \longrightarrow H^*(\Gamma(E); C^\infty(M, \mathbb{R})),$$

where $i : H_{dR}^*(M) \longrightarrow H_{Leib}^*(M)$ is the homomorphism induced by the natural inclusion.

Using Proposition 4.2, we have

Corollary 4.4 *Let (M, Λ) be a Nambu-Poisson manifold of order n , with $n \geq 3$, and let $(\wedge^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$ be the Leibniz algebroid associated with M . Suppose that*

$$\tilde{\#}^k : C_{Leib}^k(M) \longrightarrow C^k(\Omega^{n-1}(M); C^\infty(M, \mathbb{R}))$$

is the homomorphism defined by

$$\tilde{\#}^k(c^k)(\alpha_1, \dots, \alpha_k) = c^k(\#\alpha_1, \dots, \#\alpha_k),$$

for $c^k \in C_{Leib}^k(M)$ and $\alpha_1, \dots, \alpha_k \in \Omega^{n-1}(M)$. Then, the mappings $\tilde{\#}^k$ induce a homomorphism of complexes

$$\tilde{\#} : (C_{Leib}^*(M), d) \longrightarrow (C^*(\Omega^{n-1}(M); C^\infty(M, \mathbb{R})), \partial).$$

Therefore, we have the corresponding homomorphism in cohomology

$$\tilde{\#} : H_{Leib}^*(M) \longrightarrow H^*(\Omega^{n-1}(M); C^\infty(M, \mathbb{R})).$$

For the particular case of a Nambu-Poisson structure coming from a volume form, we deduce the following result:

Proposition 4.5 *Let M be an oriented manifold of dimension m , with $m \geq 3$, and let ν be a volume form on M . Then the Leibniz cohomology of the algebroid associated with (M, Λ_ν) is isomorphic to the Leibniz cohomology of M .*

Proof: Since ν is a volume form, the homomorphism

$$\# : \Omega^{m-1}(M) \longrightarrow \mathfrak{X}(M),$$

defined by (2.3) and (2.4), is an isomorphism.

Using this fact and Corollary 4.4 the result follows. \square

Remark 4.6 Note that the Leibniz algebroid $(\wedge^{n-1}(T^*M), \llbracket \cdot, \cdot \rrbracket, \#)$ associated with (M, Λ_ν) is also a Lie algebroid (see Corollary 3.9) and that the Lie algebroid cohomology of $\wedge^{n-1}(T^*M)$ is isomorphic to the de Rham cohomology of M .

For a Nambu-Poisson manifold (M, Λ) of order n , we denote by $\{ \cdot, \cdot \}'$ the Leibniz bracket on $\wedge^{n-1}(C^\infty(M, \mathbb{R}))$ characterized by (2.12). Then, the real vector space $C^\infty(M, \mathbb{R})$ is a $\wedge^{n-1}(C^\infty(M, \mathbb{R}))$ -module with the multiplication given by (2.13). Thus, we can consider the corresponding differential complex $(C^*(\wedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R})), \partial')$ and its cohomology $H^*(\wedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R}))$.

In the next result, we obtain a relation between the Leibniz algebroid cohomology of $\wedge^{n-1}(T^*M)$ and the cohomology $H^*(\wedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R}))$.

Proposition 4.7 *Let (M, Λ) be a Nambu-Poisson manifold of order n , with $n \geq 3$. Then the mapping*

$$\Phi : \wedge^{n-1}(C^\infty(M, \mathbb{R})) \longrightarrow \Omega^{n-1}(M), \quad f_1 \wedge \dots \wedge f_{n-1} \mapsto df_1 \wedge \dots \wedge df_{n-1} \quad (4.3)$$

induces a natural homomorphism of complexes

$$\tilde{\Phi} : (C^*(\Omega^{n-1}(M); C^\infty(M, \mathbb{R})), \partial) \longrightarrow (C^*(\wedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R})), \partial')$$

and therefore we have the corresponding homomorphism in cohomology

$$\tilde{\Phi} : H^*(\Omega^{n-1}(M); C^\infty(M, \mathbb{R})) \longrightarrow H^*(\wedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R})).$$

Proof: Consider the mappings

$$\tilde{\Phi}^k : C^k(\Omega^{n-1}(M); C^\infty(M, \mathbb{R})) \longrightarrow C^k(\wedge^{n-1}(C^\infty(M, \mathbb{R})); C^\infty(M, \mathbb{R}))$$

defined by

$$\tilde{\Phi}^k(c^k)(F_1, \dots, F_k) = c^k(\Phi(F_1), \dots, \Phi(F_k)), \quad (4.4)$$

for $F_1, \dots, F_k \in \wedge^{n-1}(C^\infty(M, \mathbb{R}))$.

From (2.12), (3.5) and (4.3), we get

$$\Phi(\{F_i, F_j\}') = \llbracket \Phi(F_i), \Phi(F_j) \rrbracket.$$

Using this fact, (2.5), (2.10), (2.13), (4.2) and (4.4), we obtain that the mappings $\tilde{\Phi}^k$ induce a homomorphism of complexes. \square

Next, we will introduce the modular class of an oriented Nambu-Poisson manifold. For this purpose, we prove the following result:

Theorem 4.8 *Let (M, Λ) be an oriented m -dimensional Nambu-Poisson manifold of order n , with $n \geq 3$, and ν be a volume form. Consider the mapping*

$$\mathcal{M}_\nu : C^\infty(M, \mathbb{R}) \times \dots^{(n-1)} \dots \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R}),$$

defined by

$$\mathcal{L}_{X_{f_1 \dots f_{n-1}}} \nu = \mathcal{M}_\nu(f_1, \dots, f_{n-1}) \nu \quad (4.5)$$

for $f_1, \dots, f_{n-1} \in C^\infty(M, \mathbb{R})$. Then:

(i) \mathcal{M}_ν is a skew-symmetric $(n-1)$ -linear mapping and a derivation in each argument with respect to the usual product of functions. Thus, \mathcal{M}_ν defines an $(n-1)$ -vector on M .

(ii) The mapping

$$\mathcal{M}_\Lambda : \Omega^{n-1}(M) \longrightarrow C^\infty(M, \mathbb{R}) \quad \alpha \mapsto i(\alpha) \mathcal{M}_\nu \quad (4.6)$$

*defines a 1-cocycle in the Leibniz cohomology complex associated to the Leibniz algebroid $(\wedge^{n-1}(T^*M), [\ , \], \#)$.*

(iii) *The cohomology class $[\mathcal{M}_\Lambda] \in H^1(\Omega^{n-1}(M); C^\infty(M, \mathbb{R}))$ does not depend on the chosen volume form.*

Proof: (i) It follows using (2.1), (2.2), (2.5), (2.6) and (4.5).

(ii) We will show that for all $\alpha \in \Omega^{n-1}(M)$ we have:

$$\mathcal{L}_{\# \alpha} \nu = [i(\alpha) \mathcal{M}_\nu + (-1)^{n-1} (i(d\alpha) \Lambda)] \nu. \quad (4.7)$$

Indeed, suppose that $\alpha = f df_1 \wedge \dots \wedge df_{n-1}$, with $f, f_1, \dots, f_{n-1} \in C^\infty(M, \mathbb{R})$.

A direct computation proves that

$$\begin{aligned} \mathcal{L}_{\# \alpha} \nu &= df \wedge i_{X_{f_1 \dots f_{n-1}}} \nu + f \mathcal{M}_\nu(f_1, \dots, f_{n-1}) \nu \\ &= df \wedge i_{X_{f_1 \dots f_{n-1}}} \nu + (i(\alpha) \mathcal{M}_\nu) \nu. \end{aligned} \quad (4.8)$$

Now, since $i_{X_{f_1 \dots f_{n-1}}}(df \wedge \nu) = 0$, we deduce that

$$df \wedge i_{X_{f_1 \dots f_{n-1}}} \nu = X_{f_1 \dots f_{n-1}}(f) \nu.$$

Adding this formula to (4.8) we obtain that (4.7) holds for $\alpha = f df_1 \wedge \dots \wedge df_{n-1}$. But this implies that (4.7) holds for all $\alpha \in \Omega^{n-1}(M)$.

Using (3.9), (4.7) and Theorem 3.6, we have that

$$i([\alpha, \beta]) \mathcal{M}_\nu = \mathcal{L}_{\# [\alpha, \beta]} \nu + (-1)^n (i(d[\alpha, \beta]) \Lambda) \nu = \# \alpha (i(\beta) \mathcal{M}_\nu) - \# \beta (i(\alpha) \mathcal{M}_\nu).$$

This proves (ii) (see (4.2)).

(iii) Let ν' be another volume form on M . Then there exists $f \in C^\infty(M, \mathbb{R})$, $f \neq 0$ at every point, such that $\nu' = f\nu$. We can suppose, without the loss of generality, that $f > 0$.

A direct computation, using (4.7), shows that for all $\alpha \in \Omega^{n-1}(M)$

$$i(\alpha)\mathcal{M}_{\nu'} = i(\alpha)\mathcal{M}_\nu + \# \alpha(\ln f)$$

which implies that (see (4.2))

$$\mathcal{M}_{\nu'} = \mathcal{M}_\nu + \partial(\ln f).$$

□

Theorem 4.8 allows us to introduce the following definition.

Definition 4.9 *Let (M, Λ) be an oriented Nambu-Poisson manifold of order n , with $n \geq 3$, and \mathcal{M}_Λ be the cocycle defined by (4.6). The cohomology class*

$$[\mathcal{M}_\Lambda] \in H^1(\Omega^{n-1}(M); C^\infty(M, \mathbb{R}))$$

is called the modular class of (M, Λ) .

Remark 4.10 Definition 4.9 extends for Nambu-Poisson manifolds of order greater than 2 the notion of modular class of a Poisson manifold introduced by Weinstein [29] (see also [4]).

For a Nambu-Poisson structure induced by a volume form, we deduce:

Proposition 4.11 *Let M be an oriented m -dimensional manifold and ν a volume form on M . Then the modular class of (M, Λ_ν) is null.*

Proof: Using (2.6), (2.7) and (2.8), we obtain that

$$\mathcal{L}_{X_{f_1 \dots f_{n-1}}} \nu = 0,$$

for all $f_1, \dots, f_{n-1} \in C^\infty(M, \mathbb{R})$. This implies that $\mathcal{M}_\nu = 0$ and therefore, $\mathcal{M}_{\Lambda_\nu} = 0$. □

Remark 4.12 Suppose that N and L are oriented manifolds and let ν be a volume form on N . The Nambu-Poisson structure Λ_ν on N induces a Nambu-Poisson structure Λ on the product manifold $M = N \times L$ (see Examples 2.1) and from Proposition 4.11, it follows that the modular class of (M, Λ) is null.

Using Theorem 2.2 and Remark 4.12, we have the following.

Corollary 4.13 *Let M be an oriented m -dimensional Nambu-Poisson manifold of order n , with $3 \leq n \leq m$. If at a point $x \in M$ we have $\Lambda(x) \neq 0$, then there exists an open neighborhood U of x in M such that the modular class of (U, Λ_U) is null. Here Λ_U denotes the Nambu-Poisson structure induced by Λ on U .*

The above results and the following example show that the vanishing of the modular class of a Nambu-Poisson manifold is closely related with its regularity.

Example 4.14 Consider on \mathbb{R}^3 the 3-vector defined by

$$\Lambda = x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3},$$

where (x^1, x^2, x^3) denote the usual coordinates on \mathbb{R}^3 .

The 3-vector Λ defines a Nambu-Poisson structure of order 3 on \mathbb{R}^3 .

Let ν be the volume form given by

$$\nu = dx^1 \wedge dx^2 \wedge dx^3.$$

A direct computation proves that

$$X_{x^1 x^2} = x^3 \frac{\partial}{\partial x^3}, \quad X_{x^1 x^3} = -x^3 \frac{\partial}{\partial x^2}, \quad X_{x^2 x^3} = x^3 \frac{\partial}{\partial x^1},$$

and

$$\mathcal{L}_{X_{x^1 x^2}} \nu = \nu, \quad \mathcal{L}_{X_{x^1 x^3}} \nu = \mathcal{L}_{X_{x^2 x^3}} \nu = 0.$$

Thus, $\mathcal{M}_\nu = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$.

Now, if the class modular of (\mathbb{R}^3, Λ) would be null then there exists $f \in C^\infty(\mathbb{R}^3, \mathbb{R})$ such that

$$i(\alpha) \mathcal{M}_\nu = \partial f(\alpha),$$

for all $\alpha \in \Omega^2(\mathbb{R}^3)$. Taking $\alpha = dx^1 \wedge dx^2$, we would deduce that

$$1 = X_{x^1 x^2}(f) = x^3 \frac{\partial f}{\partial x^3}.$$

But this is not possible. Thus $[\mathcal{M}_\Lambda] \neq 0$.

Acknowledgments

This work has been partially supported through grants DGICYT (Spain) (Projects PB97-1257 and PB97-1487) and Project U.P.V. 127.319-EA043/97. ML and JCM wish to express his gratitude for the heartly hospitality offered to them in the Department of Fundamental Mathematics (University of La Laguna) and in the Department of Mathematics (University of Basque Country) respectively, where part of this work was done.

References

- [1] D. Alekseevsky, P. Guha: On decomposability of Nambu-Poisson tensor, *Acta Math. Univ. Comenianae*, **65** (1996), 1-10.
- [2] F. Bayen, M. Flato: Remarks concerning Nambu's generalized mechanics, *Physical Review D*, **11** (1975), 3049-3053.
- [3] K.H. Bhaskara, K. Viswanath: Calculus on Poisson manifolds, *Bull. London Math. Soc.*, **20** (1988), 68-72.
- [4] J.L. Brylinski, G. Zuckerman: The outer derivation of a complex Poisson manifold, *J. reine angew. Math.*, **506** (1999), 181-189.
- [5] R. Chatterjee: Dynamical symmetries and Nambu mechanics, *Lett. Math. Phys.*, **36** (1996), 117-126.
- [6] R. Chatterjee, L.A. Takhtajan: Aspects of classical and quantum Nambu mechanics, *Lett. Math. Phys.*, **37**, 4 (1996), 475-482.
- [7] Y.L. Daletski, L.A. Takhtajan: Leibniz and Lie algebra structures for Nambu algebra, *Lett. Math. Phys.*, **39** (1997), 127-141.
- [8] J.-P. Dufour, A. Haraki: Rotationnels et structures de Poisson quadratiques, *C.R. Acad. Sci. Paris*, **312** (1991), 137-140.
- [9] S. Evens, J.-H. Lu, A. Weinstein: Transverse measures, the modular class, and a cohomology pairing for Lie algebroids, *Quart. J. Math. Oxford*, to appear.
- [10] M. Flato, G. Dito, D. Sternheimer: Nambu mechanics, n -ary operations and their quantization, in *Deformation Theory and Symplectic Geometry*, Mathematical Physics Studies, vol. 20. Kluwer Acad. Publ., Dordrecht (1997), 43-66.
- [11] Ph. Gautheron: Some remarks concerning Nambu mechanics, *Lett. Math. Phys.*, **37** (1996), 103-116.
- [12] J. Grabowski, G. Marmo: Remarks on Nambu-Poisson and Nambu-Jacobi brackets, *J. Phys. A: Math. Gen.*, **32** (1999), 4239-4247.
- [13] J. Grabowski, G. Marmo: On Filippov algebroids and multiplicative Nambu-Poisson structures, *preprint math.DG/9902127. Diff. Geom. and its appl.*, to appear.
- [14] R. Ibáñez, M. de León, J.C. Marrero, D. Martín de Diego: Dynamics of generalized Poisson and Nambu-Poisson brackets, *J. Math. Phys.*, **38** (5) (1997), 2332-2344.
- [15] A. Lichnerowicz: Les variétés de Poisson et leurs algèbres de Lie associées, *J. Differential Geometry*, **12** (1977), 253-300.

- [16] Z.J. Liu, P. Xu: On quadratic Poisson structures, *Lett. Math. Phys.*, **26** (1992), 33-42.
- [17] J.L. Loday: *Cyclic Homology*, Grund. Math. Wissen. **301**, Springer Verlag (1992).
- [18] J.L. Loday: Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *L'Enseignement Math.*, **39** (1993), 269-293.
- [19] J.L. Loday, T. Pirashvili: Universal enveloping algebras of Leibniz and (co)-homology, *Math. Ann.*, **296** (1993), 139-158.
- [20] J.M. Lodder: Leibniz cohomology for differentiable manifolds, *Ann. Inst. Fourier, Grenoble* **48** (1998), 73-95.
- [21] K. Mackenzie: *Lie groupoids and Lie algebroids in differential geometry*, London Math. Soc. Lecture Notes Series, **124** Cambridge Univ. Press, Cambridge (1987).
- [22] G. Marmo, G. Vilasi, A.M. Vinogradov: The local structure of n -Poisson and n -Jacobi manifolds, *J. Geom. Phys.*, **25** (1998), 141-182.
- [23] N. Nakanishi: On Nambu-Poisson manifolds, *Rev. Math. Phys.*, **10** (1998), 499-510.
- [24] Y. Nambu: Generalized Hamiltonian dynamics, *Physical Review D*, **7**, (8) (1973), 2405-2412.
- [25] L. Takhtajan: On foundations of the generalized Nambu mechanics, *Commun. Math. Phys.*, **160** (1994), 295-315.
- [26] I. Vaisman: *Lectures on the Geometry of Poisson Manifolds*, Progress in Math. 118, Birkhäuser, Basel (1994).
- [27] I. Vaisman: Nambu-Lie groups, *preprint math.DG/9812064*.
- [28] A. Weinstein: The local structure of Poisson manifolds, *J. Differential Geometry*, **18** (1983), 523-557. Errata et addenda: *J. Differential Geometry*, **22** (1985), 255.
- [29] A. Weinstein: The modular automorphism group of a Poisson manifold, *J. Geom. Phys.*, **23** (1997), 379-394.
- [30] A. Weinstein: Poisson geometry, *Differential Geometry and its Applications*, **9** (1998), 213-238.
- [31] P. Xu: Gerstenhaber algebras and BV-algebras in Poisson geometry, *Commun. Math. Phys.*, **200** (1999), 545-560.